Proximity in the Age of Distraction: Robust Approximate Nearest Neighbor Search

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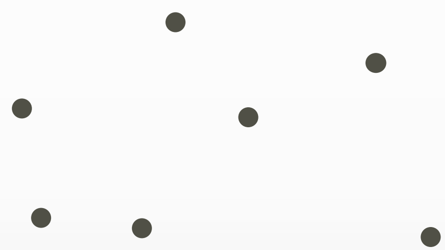
MIT

Nearest Neighbor Problem

Nearest Neighbor

Dataset of n points P in a metric space (X, d_X), e.g. $\mathbb{R}^d$



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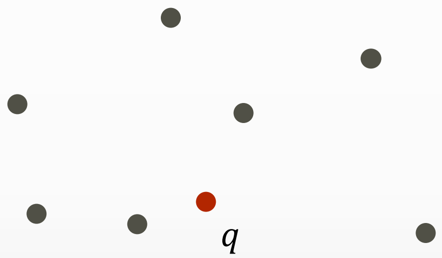

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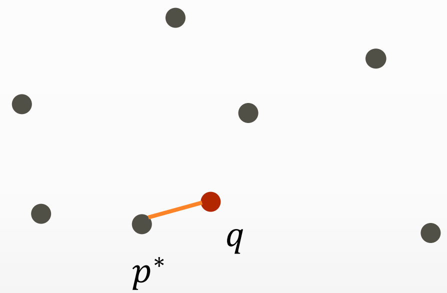

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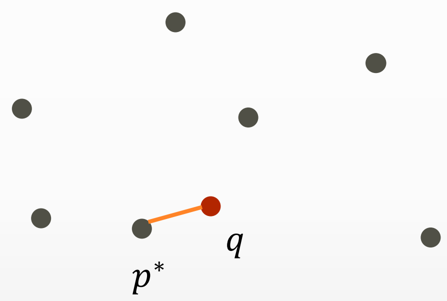

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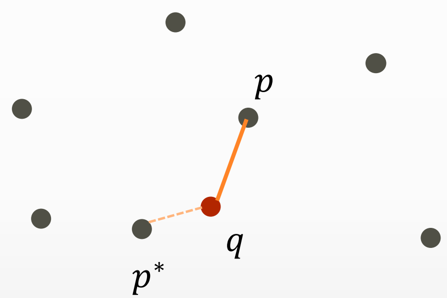

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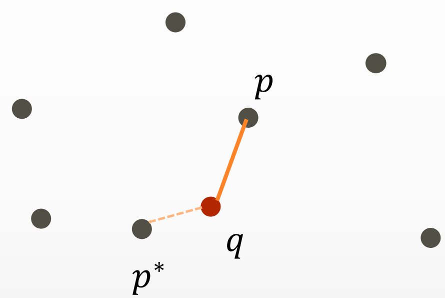

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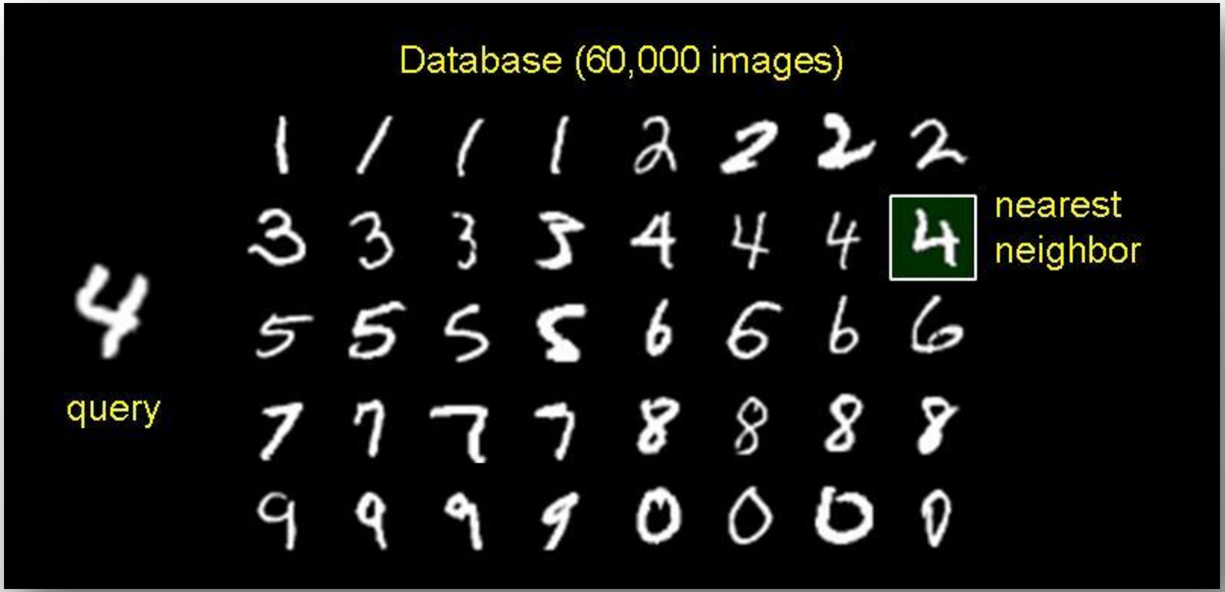
  - If optimal distance is r, report a point in distance cr for $c = (1 + \epsilon)$
  - For Hamming (and L_1) query time is $n^{1/O(c)}$ [IM98]
  - and for Euclidean (L_2) it is $n^{\frac{1}{O(c^2)}}$ [AI08]

Applications of NN

Searching for the closest object



Robust NN Problem

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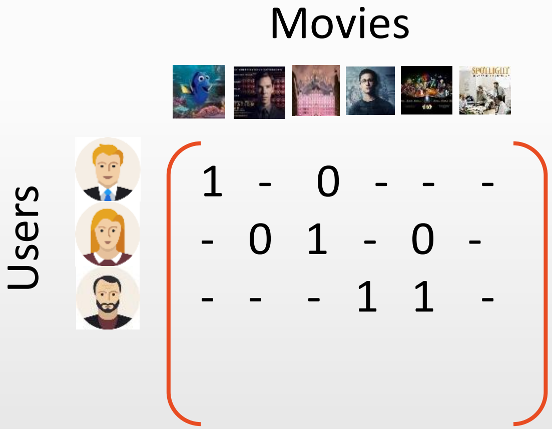

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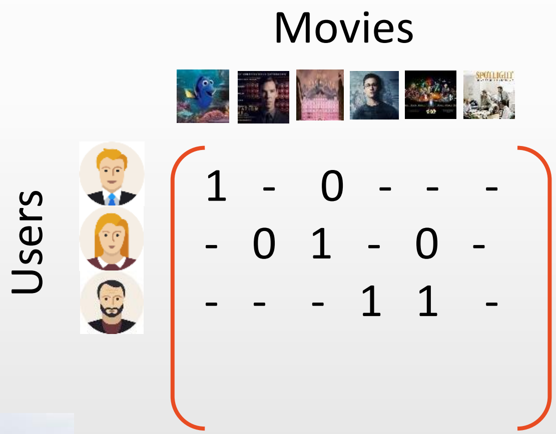

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- Irrelevant
  - Occluded image


The Robust NN problem

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$$p_1 = (3,4,0,5)$$
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- ➢ Different set of coordinates for different points
- ➢ Applying this naively would require $\binom{d}{k} \approx d^k$

Budgeted Version

- Dataset of n points P in $\mathbb{R}^d$
- d weights
  $w = (w_1, w_2, \dots, w_d) \in [0,1]^d$

$w = (0.5, 0.5, 0.8, 0.3)$

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Why not single criterion?

- Equivalent to exact near neighbor in Hamming: there is a point within distance r of the query iff there is a point within distance 0 after ignoring $k = r$ coordinates

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$(1+\epsilon)$-approximation	$r(1+\epsilon)$	$O(\frac{k}{\epsilon\delta})$	$O(\frac{n^{\delta}}{\epsilon})$	$(1+\epsilon)$ −ANN

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Budgeted Version	$O(r)$	Weight of $O(1)$	n^{δ} $+O(n^{\delta}d^4)$	2-ANN

Algorithm

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Theorem. If for a point $p^* \subset P$, the L_1 distance of q and p^* is at most r after removing k coordinates, there exists an algorithm which reports a point p whose distance to q is $O(r/\delta)$ after removing $O(k/\delta)$ coordinates.

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- A set of randomized maps $\boldsymbol{f_1, f_2, \ldots f_m}: \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$
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- W.l.o.g. assume that the query is the origin
  - Find the data point with minimum norm.

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- **Far point**
  - v has at least k/δ large coordinates
  - Probability of missing large coordinates is **at most** $\left(1 - \frac{\delta}{k}\right)^{(k/\delta) \cdot \ln n} \approx 1/n$

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- Repeat this embedding $\mathrm{O}(n^\delta \log n)$ times and report the best.

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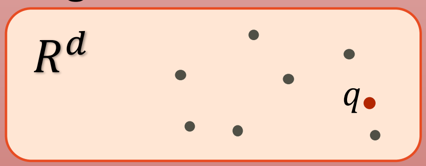

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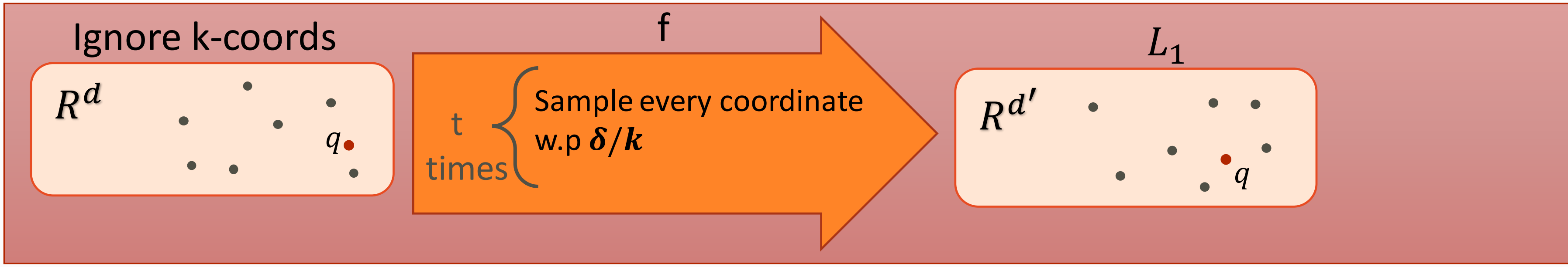

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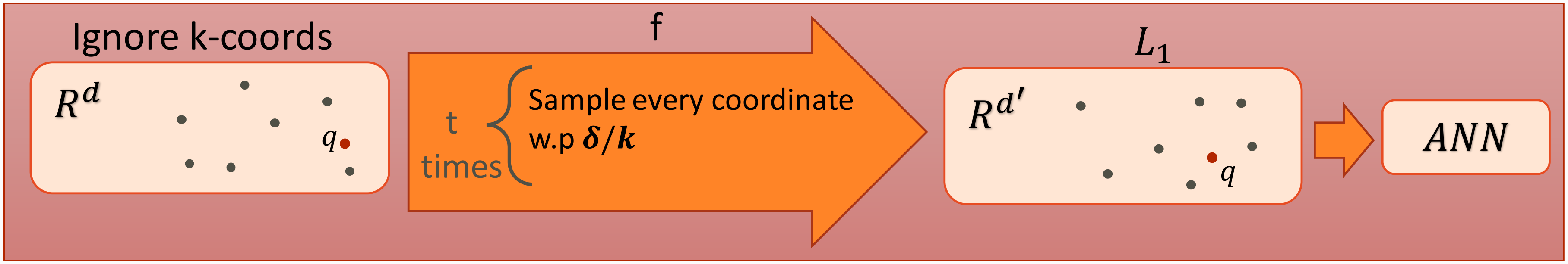

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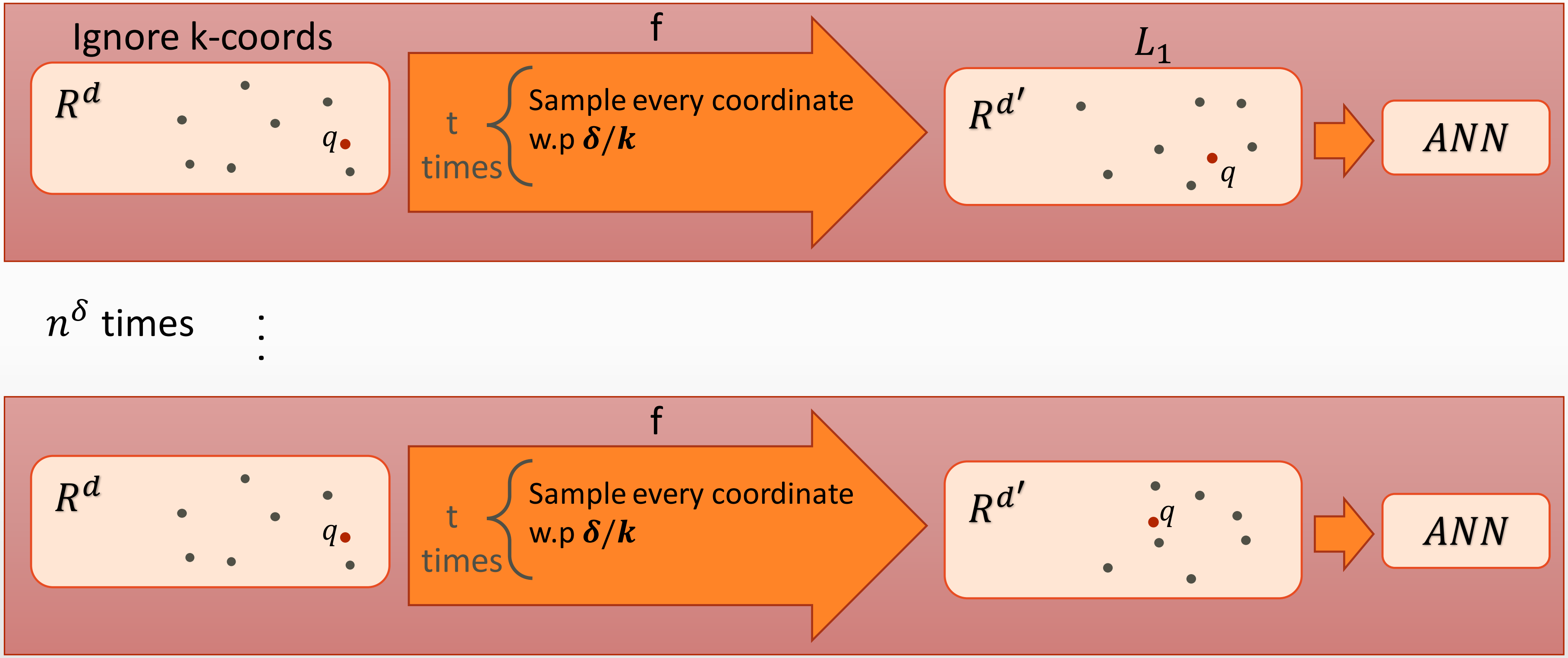
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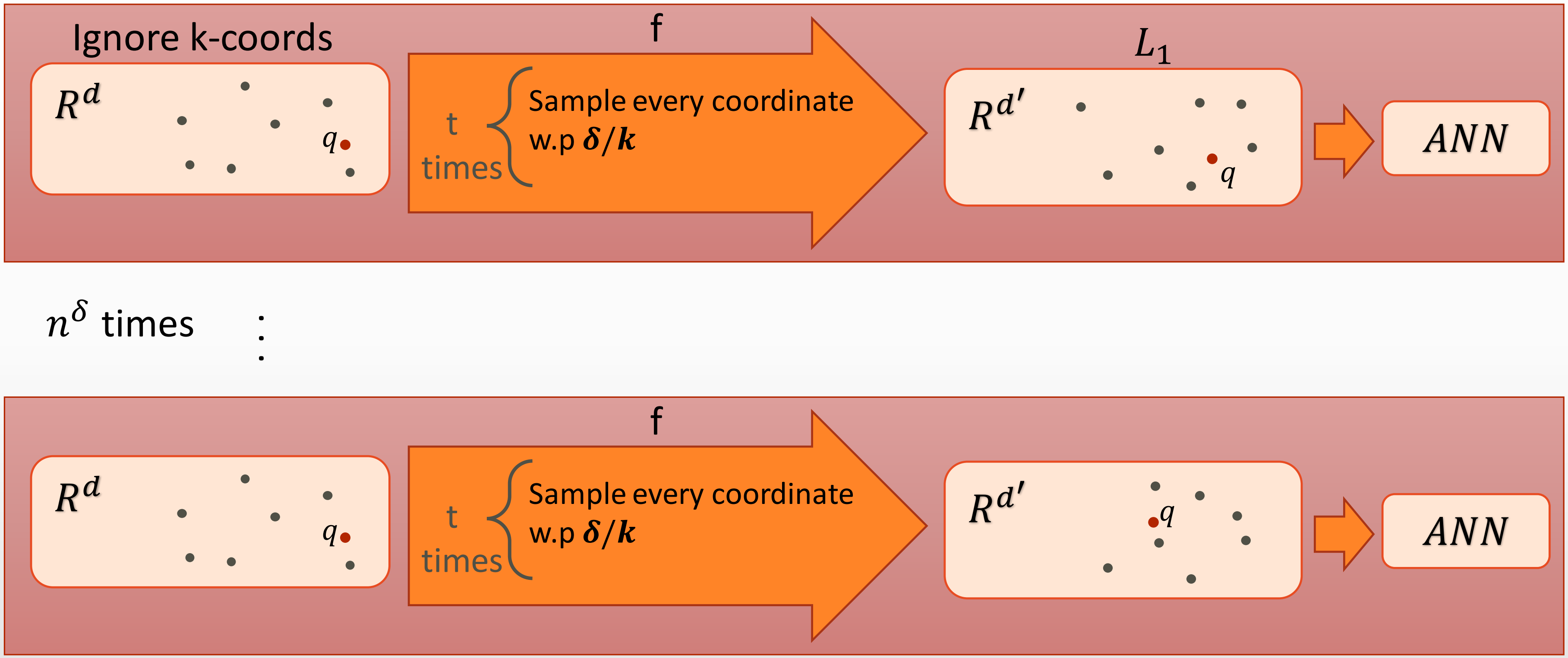

Check the distance of all n^δ candidates and report the closest one after ignoring k coordinates

Analysis

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Claim:

- A close point is $2r$ −light.

Truncation

Coordinates are not necessarily $\mathbf{0}$ and ∞

Let $\boldsymbol{f'}$ be obtained by sampling every coordinate with probability $\boldsymbol{\tau = \delta/k}$

- $\mathbb{E}[|f'(v)|_1] = \tau|v|_1$
- $\mathbf{Var}[|f'(v)|_1] = \tau\,(1-\tau)|v|_2^2$

Need to **bound the influence** of every coordinate.

- Truncate every coordinate at $\boldsymbol{r/k}$,i.e., $\boldsymbol{v_i = \min\{v_i, r/k\}}$
- $\boldsymbol{r}$ **−Light point:** a point with norm $\leq r$ after truncation
- **R− Heavy point:** a point with norm $\geq R$ after truncation
- **Close point:** a point with norm $\leq r$ after ignoring k coordinates
- **Far point:** a point with norm $\geq r/\delta$ after ignoring k/δ coordinates

Claim:

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➢ **Analyze the behavior of the maps over the truncated points instead.**

L_1 Norm

Using truncation

- Bound the variance and prove concentration for $\boldsymbol{f'}$ by Chebyshev

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f **is a concatenation of** $t = O(\ln n)$ **such** f'

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$\boldsymbol{f}$ is a concatenation of $\boldsymbol{t = O(\ln n)}$ such $\boldsymbol{f'}$

- $\mathbb{E}[|f(v)|_1] = t\tau|v|_1$
- Prove concentration for $\boldsymbol{f}$ using Chernoff

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	distance	#ignored coordinates	Query Time #Queries	Query Time Query type
Opt	r	k		
L_1	$O(\frac{r}{\delta})$	$O(\frac{k}{\delta})$	n^δ	2-ANN

Generalizations

- L_p norm

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- **L_p norm**
  - Minimize the $|v|_p^p$ norm, i.e., $\sum_i v_i^p$ similar to the L_1 norm

Generalizations

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- Minimize the $|\boldsymbol{v}|_{\boldsymbol{p}}^{\boldsymbol{p}}$ norm, i.e., $\sum_{\boldsymbol{i}} \boldsymbol{v}_{\boldsymbol{i}}^{\boldsymbol{p}}$ similar to the L_1 norm

$L_{\boldsymbol{p}}$	$O(r\left(c+\frac{1}{\delta}\right)^{1/p})$	$O(k\left(c+\frac{1}{\delta}\right))$	n^{δ}	$c^{1/\mathrm{p}}$**-ANN**

Generalizations

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- **Map:**
  - sample coordinate i with probability proportional to $1/w_i$

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- **Truncation:**
  - Truncate coordinate i with by value $\frac{r}{c/w_i - 1}$

Generalizations

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L_p	$O(r\left(c+\frac{1}{\delta}\right)^{1/p})$	$O(k\left(c+\frac{1}{\delta}\right))$	n^{δ}	$c^{1/p}$-ANN

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    - E.g. a coordinate of cost approaching 0 will be truncated to 0

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- ❑ **Budgeted**

- **Map:**
  - sample coordinate $\boldsymbol{i}$ with probability proportional to $\mathbf{1}/\boldsymbol{w}_{\boldsymbol{i}}$
  - To maintain the expectation multiply sampled coordinates by $\boldsymbol{w}_{\boldsymbol{i}}$
- **Truncation:**
  - Truncate coordinate $\boldsymbol{i}$ with by value $\frac{\boldsymbol{r}}{\boldsymbol{c}/\boldsymbol{w}_{\boldsymbol{i}}-\mathbf{1}}$
  - E.g. a coordinate of cost approaching 0 will be truncated to 0

Budgeted Version	$O(r)$	Weight of $O(1)$	n^{δ} $+O(n^{\delta}d^4)$	**2-ANN**

Conclusion

	distance	#ignored coordinates	Query Time #Queries	Query Time Query type
Opt	r	k		
L_1	$O(\frac{r}{\delta})$	$O(\frac{k}{\delta})$	n^{δ}	2-ANN
L_p	$O(r\left(c+\frac{1}{\delta}\right)^{1/\mathbf{p}})$	$O(k\left(c+\frac{1}{\delta}\right))$	n^{δ}	$c^{1/\mathbf{p}}$-ANN
$(1+\epsilon)$-approximation	$r(1+\epsilon)$	$O(\frac{k}{\epsilon\delta})$	$O(\frac{n^{\delta}}{\epsilon})$	$(1+\epsilon)$ −ANN
Budgeted Version	$O(r)$	Weight of $O(1)$	n^{δ} $+O(n^{\delta}d^{4})$	2-ANN

Conclusion

	distance	#ignored coordinates	Query Time	
			#Queries	Query type
Opt	r	k		
L_1	$O(\frac{r}{\delta})$	$O(\frac{k}{\delta})$	n^δ	2-ANN
L_p	$O(r\left(c+\frac{1}{\delta}\right)^{1/p})$	$O(k\left(c+\frac{1}{\delta}\right))$	n^δ	$c^{1/\mathbf{p}}$-ANN
$(1+\epsilon)$-**approximation**	$r(1+\epsilon)$	$O(\frac{k}{\epsilon\delta})$	$O(\frac{n^\delta}{\epsilon})$	$(1+\epsilon)$ −ANN
Budgeted Version	$O(r)$	Weight of $O(1)$	$n^\delta + O(n^\delta d^4)$	2-ANN

Open Problems

- Improve the dependence on δ
- Prove lower bounds

Conclusion

	distance	#ignored coordinates	Query Time #Queries	Query type
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L_1	$O(\frac{r}{\delta})$	$O(\frac{k}{\delta})$	n^δ	2-ANN
L_p	$O(r\left(c+\frac{1}{\delta}\right)^{1/p})$	$O(k\left(c+\frac{1}{\delta}\right))$	n^δ	$c^{1/\mathbf{p}}$-ANN
$(1+\epsilon)$-approximation	$r(1+\epsilon)$	$O(\frac{k}{\epsilon\delta})$	$O(\frac{n^\delta}{\epsilon})$	$(1+\epsilon)$ −ANN
Budgeted Version	$O(r)$	Weight of $O(1)$	n^δ $+O(n^\delta d^4)$	2-ANN

Open Problems

- Improve the dependence on δ
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Thank You!